A SOLUTION OF THE QUANTUM KNIZHNIK ZAMOLODCHIKOV EQUATION OF TYPE C_n

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Abstract. We construct a solution of Cherednik's quantum Knizhnik Zamolodchikov equation associated with the root system of type C_n . This solution is given in terms of a restriction of a q-Jordan-Pochhammer integral. As its applicaton, we give an explicit expression of a special case of the Macdonald polynomial of the C_n type. Finally we explain the connection with the representation of the Hecke algebra.

1. Introduction

We study the quantum Knizhnik Zamolodchikov (QKZ) equation ([2]) associated with the root system of type C_n. A solution to this equation is found by means of a restriction of the q-Jordan-Pochhammer integral.

A solution of the QKZ equation of type A_{n-1} is given in [14]. Since the appearance of that work, however, there has been no progress in the study of the QKZ equation for other types of root systems with regard to the determination of solutions. This paper is devoted to such a task.

To construct our solution, we exploit a family of rational functions which would correspond to a basis of the q de Rham cohomology attached to the integrand. This turns out to be a natural basis for the representation of the Hecke algebra $H(W)$ through the Lusztig operator T_i .

Next, as a byproduct of our investigation, we obtain an integral representaion of the special case of an eigenfunction associated with the Macdonald operator of the C_n type. In particular, it is seen that, taking a suitable cycle, a restriction of the q-Jordan-Pochhammer integral expresses the Macdonald polynomial of the C_n type parametrized by the partition $(\lambda, 0, \dots, 0)$. This integral leads to a more explicit expression.

We believe that the present paper represents a first step toward understanding the BC_n type QKZ equation and the BC_n type Macdonald polynomial. It is noteworthy that even in the classical (q=1) case was not previously known that such an integral gives spherical functions associated with the root system C_n. For related works on BC_n type spherical functions, we refer the reader to [6] and references therein.

Throughout this paper, q is regarded as a real number satisfying $0 \leq q < 1$.

2. QKZ equation of type C_n

We first give a review of the QKZ equation associated with the root system of type C_n for the reader's convenience, following Cherednik [2] and Kato [7].

Key words and phrases. Quantum Knizhnik Zamolodchikov equations, Macdonald polynomials, q-Jordan-Pochhammer integrals.

Let $E = \oplus_{1\le i\le n}\mathbb{R}\epsilon_i$ be the real Euclidean space with inner product $\langle\,,\,\rangle$ such that $\langle \epsilon_i, \epsilon_j\rangle = \delta_{ij}$. Let $\Delta = \{\, \pm\epsilon_i \pm \epsilon_j\,(\,1\le i<j\le n\,), \pm 2\epsilon_i\,(\,1\le i\le n\,)\,\}$ be the root system of type C_n, $\Delta^+ = \{\, \epsilon_i \pm \epsilon_j\,(\,1\le i<j\le n\,), 2\epsilon_i\,(\,1\le i\le n\,)\,\}$ the set of positive roots, $\Pi = \{\, \alpha_i = \epsilon_i - \epsilon_{i+1}\,(\,1\le i\le n-1\,),\ \alpha_n = 2\epsilon_n\,\}$ the set of simple roots, $P = \oplus_{1\le i\le n}\,\mathbb{Z}\epsilon_i$ the weight lattice, and $P^\vee = \oplus_{1\le i\le n}\,\mathbb{Z}\epsilon_i + \mathbb{Z}(\frac{1}{2}\sum_{i=1}^n \epsilon_i)$ the dual weight lattice for the root system Δ. We frequently write $\alpha\in\Delta^+$ as $\alpha>0$.

An element of the group algebra $A = \mathbb{C}[P]$ is denoted by e^λ , as is customary. Then the Weyl group $W = W(C_n) = \langle\, s_1, s_2, \dots, s_n\,\rangle$ (where each s_i is a standard generator corresponding to the simple root α_i) acts on A as $w(e^\lambda) = e^{w\lambda}$ $(w\in W)$. The symbol s_α denoting the reflections is defined by $s_\alpha(x) = x - \langle x,\alpha\rangle\alpha^\vee$, with $\alpha^\vee = 2\alpha/\langle\,\alpha,\alpha\rangle$ for $x\in E$ and $\alpha\in\Delta$.

The set of affine roots associated with Δ is $\tilde\Delta = \{\,\alpha+m\delta\,;\ \alpha\in\Delta,\ m\in\mathbb{Z}\}$, where δ denotes the constant function 1 on E. The simple roots are $a_0 = -\theta+\delta$ with the highest root $\theta = 2\epsilon_1$ and $a_i = \alpha_i\in\Delta$ for $1\le i\le n$. We use the symbol introduced above, $s_i\,(0\le i\le n)$ to also represent the generator for the corresponding affine Weyl group. We note that $s_0 = \tau(\theta^\vee)s_\theta = \tau(\epsilon_1)s_{2\epsilon_1}$, where $\tau(\mu)$ is a translation by μ .

Let us introduce V as the left free $A-$module of rank $|W| = 2^n\,n!$ with the free basis $h_w\,(w\in W)$; each element F of V can be written uniquely as $F = \sum_{w\in W} f_w h_w$ $(f_w\in A)$. Then, let $A^\sim$ be a completion of the quotient field of A . We then have $V^\sim = A^\sim\otimes_A V$. The action r_w of the Weyl group W on $V^\sim$ is defined by the following:

$$r_w(fh_y) = w(f)h_{wy} \quad \text{for} \quad f\in A \quad \text{and} \quad w,y\in W\,.$$

Moreover, the action of the translation $\tau(\mu)\,(\mu\in P^\vee)$ for a parameter $u\in E$ is given by

$$\tau(\mu)e^\lambda = q^{-\langle\,\lambda,\mu\,\rangle}e^\lambda \quad \text{for} \quad \lambda\in P\,, \quad \tau(\mu)h_w = q^{\langle\,\mu,wu\,\rangle}h_w \quad \text{for} \quad w\in W$$

and

$$r_{\tau(\mu)}(fh_w) = \tau(\mu)(f)q^{\langle\,\mu,wu\,\rangle}h_w \quad \text{for} \quad f\in A \quad \text{and} \quad w\in W\,.$$

This is an evaluation representation for which e^δ is identified with q .

Hereafter the symbol r_w is used also to represent the element w from the extended affine Weyl group $W_{P^\vee} = W\ltimes P^\vee$ (the semidirect product of W and $P^\vee$). Then $r_w\,\tau(\epsilon_i) = \tau(w(\epsilon_i))\,r_w$. Note also that, if $w = v\tau(\lambda),\ v\in W,\ \lambda\in P^\vee$, we have $w(\mu) = v\mu - \langle\,\lambda,\mu\,\rangle\,\delta$ for $\mu\in P$.

For an affine root $\alpha+m\delta$ $(\alpha\in\Delta\,, m\in\mathbb{Z})$, define the R-matrix $R_{\alpha+m\delta}$ as an element of $End_{A^\sim}(V^\sim)$ by the formula

$$R_{\alpha+m\delta}h_y = \begin{cases} a_{\alpha+m\delta}h_y + q^{m\,\langle\,\alpha^\vee\,,\,yu\,\rangle}b_{\alpha+m\delta}h_{s_\alpha y}\,, & y^{-1}(\alpha)>0\,,\\ c_{\alpha+m\delta}h_y + q^{m\,\langle\,\alpha^\vee\,,\,yu\,\rangle}d_{\alpha+m\delta}h_{s_\alpha y}\,, & y^{-1}(\alpha)<0 \end{cases}$$

for $y\in W$, where

$$a_{\alpha+m\delta} = \frac{1-q^m e^\alpha}{1-t_\alpha q^m e^\alpha}\,, \qquad b_{\alpha+m\delta} = \frac{1-t_\alpha}{1-t_\alpha q^m e^\alpha}\,,$$

$$c_{\alpha+m\delta} = \frac{t_\alpha(1-q^m e^\alpha)}{1-t_\alpha q^m e^\alpha}\,, \qquad d_{\alpha+m\delta} = \frac{q^m e^\alpha(1-t_\alpha)}{1-t_\alpha q^m e^\alpha}$$

and $\alpha \mapsto t_\alpha$ is a W-invariant function taking positive values; there are two different t_α, which we may write as $t_1 = t_{\pm\epsilon_i\pm\epsilon_j}, t_2 = t_{\pm 2\epsilon_j}$.
It is seen that

$$r_w R_\alpha = R_{w(\alpha)} r_w \quad \text{for} \quad \alpha \in \tilde{\Delta}\,, w \in W_{P^\vee}\,, \tag{2.1}$$

$$R_\beta = R_{-\beta}^{-1} \quad \text{for} \quad \beta \in \tilde{\Delta} \tag{2.2}$$

and

$$\begin{cases} R_{\epsilon_i-\epsilon_j} R_{\epsilon_i-\epsilon_k} R_{\epsilon_j-\epsilon_k} = R_{\epsilon_j-\epsilon_k} R_{\epsilon_i-\epsilon_k} R_{\epsilon_i-\epsilon_j}\,, \quad 1 \le i < j < k \le n\,, \\ R_{\epsilon_i-\epsilon_j} R_{2\epsilon_i} R_{\epsilon_i+\epsilon_j} R_{2\epsilon_j} = R_{2\epsilon_j} R_{\epsilon_i+\epsilon_j} R_{2\epsilon_i} R_{\epsilon_i-\epsilon_j}\,, \quad 1 \le i < j \le n\,. \end{cases} \tag{2.3}$$

The relations in (2.3) constitute the Yang-Baxter equation associated with the root system of type C_n.

Then we can state the definition of the QKZ equation for the root system of type C_n.

Definition 2.1. *The QKZ equation for the root system C_n with a parameter $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ is the following system of equations:*

$$r_{\tau(\epsilon_i)}^{-1} F = R_{\tau(\epsilon_i)} F\,, \qquad 1 \le i \le n,$$

and

$$r_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}^{-1} F = R_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))} F\,,$$

for $F \in V^\sim$, with

$$\begin{aligned} R_{\tau(\epsilon_i)} &= R_{\epsilon_i-\epsilon_{i-1}+\delta} \cdots R_{\epsilon_i-\epsilon_1+\delta} R_{2\epsilon_i+\delta} R_{\epsilon_1+\epsilon_i} \cdots R_{\epsilon_{i-1}+\epsilon_i} \\ &\times R_{\epsilon_i+\epsilon_{i+1}} \cdots R_{\epsilon_i+\epsilon_n} R_{2\epsilon_i} R_{\epsilon_i-\epsilon_n} \cdots R_{\epsilon_i-\epsilon_{i+1}} \end{aligned}$$

for $1 \le i \le n$, and

$$\begin{aligned} R_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))} &= (R_{2\epsilon_1} R_{\epsilon_1+\epsilon_2} R_{\epsilon_1+\epsilon_3} \cdots R_{\epsilon_1+\epsilon_n}) \\ &\times (R_{2\epsilon_2} R_{\epsilon_2+\epsilon_3} \cdots R_{\epsilon_2+\epsilon_n}) \times \cdots \times (R_{2\epsilon_{n-1}} R_{\epsilon_{n-1}+\epsilon_n}) R_{2\epsilon_n}\,. \end{aligned}$$

Remark. If we introduce the operators L_μ ($\mu \in P^\vee$) and $P_\mu^u \in End_{A^\sim}(V^\sim)$ ($\mu \in P^\vee$, $u \in E$) defined by

$$L_\mu(\sum f_w h_w) = \sum L_\mu(f_w) h_w \quad \text{with} \quad L_\mu(e^\lambda) = q^{\langle \mu, \lambda \rangle} e^\lambda \quad (\lambda \in P)$$

and

$$P_\mu^u(h_w) = q^{\langle \mu, wu \rangle} h_w\,,$$

then the equation above can be rewritten as

$$L_{\epsilon_i} F = P_{\epsilon_i}^u R_{\tau(\epsilon_i)} F$$

and

$$L_{\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)} F = P_{\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n)}^u R_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))} F\,.$$

Fulfilment of the compatibility condition of the QKZ equation is guaranteed by the Yang-Baxter equation (2.3).

In the next section, we will construct a solution of the QKZ equation for the special case $u = -\lambda\epsilon_1$ $(\lambda > 0)$ through application of the q-Jordan-Pochhammer integral.

3. Integrals and main result

We introduce the form

$$\Phi = x^\lambda \prod_{1\le j\le n} \frac{(ty_j/x)_\infty (ty_j^{-1}/x)_\infty}{(y_j/x)_\infty (y_j^{-1}/x)_\infty} \frac{dx}{x}\,, \tag{3.1}$$

where $(a)_\infty = \prod_{s\ge 0}(1-aq^s)$. This can be regarded as a form of a restriction of the q-Jordan-Pochhammer integral

$$x^\lambda \prod_{1\le j\le 2n} \frac{(ty_j/x)_\infty}{(y_j/x)_\infty} \frac{dx}{x}\,,$$

which is studied in [14] and [1].

Next, to construct our solution in case of $u = -\lambda\epsilon_1$ $(\lambda > 0)$, we use the induced representaion of the Weyl group $W = W(C_n)$ from the trivial representation of a parabolic subgroup.

As a parabolic subgroup of W, we choose a stabilizer $W_{\epsilon_1} = \langle s_2, \ldots, s_n\rangle$ of ϵ_1. A representative of the quotient W/W_{ϵ_1} is fixed to be

$$\begin{cases} w_1 = e,\ w_2 = s_1,\ w_3 = s_2 s_1,\ \ldots,\ w_{n+1} = s_n \cdots s_2 s_1, \\ w_{n+2} = s_{n-1}w_{n+1},\ w_{n+3} = s_{n-2}s_{n-1}w_{n+1},\ \ldots,\ w_{2n} = s_1\cdots s_{n-1}w_{n+1}\,. \end{cases}$$

It is seen that the element $\overline{h}_e = \sum_{g\in W_{\epsilon_1}} h_g$ is invariant under the action of W_{ϵ_1} and the induced representation of W from $\overline{h}_e$ is given by the elements

$$\overline{h}_{w_i} = \sum_{g\in W_{\epsilon_1}} h_{w_i\, g} \quad (1 \le i \le 2n)\,.$$

Using w_i as suffices, we define the following rational functions:

$$\varphi_{w_i} = \begin{cases} \dfrac{\displaystyle\prod_{1\le\mu<i}\left(1-\frac{y_\mu^{-1}}{x}\right)}{\displaystyle\prod_{1\le\mu\le i}\left(1-t\frac{y_\mu^{-1}}{x}\right)}\,, & 1\le i\le n\,, \\ \dfrac{\displaystyle\prod_{2n-i+1<\mu\le n}\left(1-\frac{y_\mu}{x}\right)}{\displaystyle\prod_{2n-i+1\le\mu\le n}\left(1-t\frac{y_\mu}{x}\right)} \displaystyle\prod_{1\le\mu\le n}\frac{\left(1-\frac{y_\mu^{-1}}{x}\right)}{\left(1-t\frac{y_\mu^{-1}}{x}\right)}\,, & n+1\le i\le 2n\,. \end{cases}$$

Associated with the function Φ, we write

$$\langle\psi\rangle = \int_{\mathcal{C}} \psi\Phi$$

for a rational function ψ and a fixed cycle $\mathcal{C}$, and define the element Ψ by

$$\Psi = \sum_{1\le i\le 2n} \langle\varphi_{w_i}\rangle \overline{h}_{w_i}\,.$$

Then we obtain the following, which will be proven in the next section.

Proposition 3.1. $r_{a_i}\Psi = R_{a_i}\Psi \qquad \text{for} \qquad 0 \le i \le n.$

We are now in a position to state our main result.

Theorem 3.2. *The function*

$$\Psi = \sum_{1\le i\le 2n} \langle \varphi_{w_i}\rangle \overline{h}_{w_i}$$

satisfies the QKZ equation of type C_n with the parameter $u = -\lambda\epsilon_1\,(\lambda > 0)$ and $t_1 = t_2 = t$:

$$r^{-1}_{\tau(\epsilon_i)}\Psi = R_{\tau(\epsilon_i)}\Psi\,, \qquad 1 \le i \le n\,, \tag{3.2}$$

and

$$r^{-1}_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\Psi = R_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\Psi\,. \tag{3.3}$$

From this point we use the identification $y_i = e^{\epsilon_i}$ for $1 \le i \le n$.

It is seen that a system of fundamental solutions is obtained by taking suitable linearly independent cycles.

Proof. We first note

$$r^{-1}_{\tau(\epsilon_1)}\Psi = r_{s_\theta\, s_0}\Psi\,.$$

Proposition 3.1 and (2.1) imply

$$r_{s_\theta s_0}\,\Psi = r_{s_\theta} r_{s_0}\,\Psi = r_{s_\theta} R_{\alpha_0}\,\Psi = R_{s_\theta(\alpha_0)} r_{s_\theta}\,\Psi\,.$$

Applying this process repeatedly, we finally obtain

$$\begin{aligned} r_{s_\theta s_0}\,\Psi =& R_{s_\theta(\alpha_0)} R_{(s_1\cdots s_n)(s_{n-1}\cdots s_2)(\alpha_1)} R_{(s_1\cdots s_n)(s_{n-1}\cdots s_3)(\alpha_2)} \cdots R_{(s_1\cdots s_n)(\alpha_{n-1})} \\ &\times R_{(s_1\cdots s_{n-1})(\alpha_n)} \cdots R_{s_1(\alpha_2)} R_{\alpha_1}\,\Psi \\ =& R_{2\epsilon_1+\delta}\, R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_1+\epsilon_n}\; R_{2\epsilon_1}\; R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_1-\epsilon_2}\,\Psi\,, \end{aligned}$$

since $s_\theta = (s_1\cdots s_{n-1})(s_n\cdots s_1)$. Thus we have

$$r^{-1}_{\tau(\epsilon_1)}\Psi = R_{2\epsilon_1+\delta}\, R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_1+\epsilon_n}\; R_{2\epsilon_1}\; R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_1-\epsilon_2}\,\Psi\,. \tag{3.4}$$

Next, let us apply $r_{s_{i-1}\cdots s_1}$ on both sides of (3.4). Then the left-hand side is

$$\begin{aligned} & r_{s_{i-1}\cdots s_1} r^{-1}_{\tau(\epsilon_1)}\,\Psi = r^{-1}_{\tau(\,s_{i-1}\cdots s_1(\epsilon_1)\,)}\, r_{s_{i-1}\cdots s_1}\Psi \\ &= r^{-1}_{\tau(\,s_{i-1}\cdots s_1(\epsilon_1)\,)}\, R_{s_{i-1}\cdots s_2(\alpha_1)}\, R_{s_{i-1}\cdots s_3(\alpha_2)} \cdots R_{s_{i-1}(\alpha_{i-2})} R_{\alpha_{i-1}}\Psi \\ &= r^{-1}_{\tau(\epsilon_i)} R_{\epsilon_1-\epsilon_i} R_{\epsilon_2-\epsilon_i} \cdots R_{\epsilon_{i-2}-\epsilon_i} R_{\epsilon_{i-1}-\epsilon_i}\Psi \\ &= R_{\epsilon_1-\epsilon_i-\delta}\, R_{\epsilon_2-\epsilon_i-\delta} \cdots R_{\epsilon_{i-2}-\epsilon_i-\delta}\, R_{\epsilon_{i-1}-\epsilon_i-\delta}\; r^{-1}_{\tau(\epsilon_i)}\Psi\,. \end{aligned}$$

This follows from the relation $\tau(-\epsilon_i)(\epsilon_j - \epsilon_i) = \epsilon_j - \epsilon_i - \delta$.

On the other hand, the right-hand side is

$$\begin{aligned} & r_{s_{i-1}\cdots s_1} R_{2\epsilon_1+\delta}\, R_{\epsilon_1+\epsilon_2} \cdots R_{\epsilon_1+\epsilon_n}\, R_{2\epsilon_1}\, R_{\epsilon_1-\epsilon_n} \cdots R_{\epsilon_1-\epsilon_2}\,\Psi \\ =& R_{2\epsilon_i+\delta}\, R_{\epsilon_1+\epsilon_i} R_{\epsilon_2+\epsilon_i} \cdots R_{\epsilon_{i-1}+\epsilon_i}\; R_{\epsilon_i+\epsilon_{i+1}} \cdots R_{\epsilon_i+\epsilon_n}\, R_{2\epsilon_i}\,\Psi\,. \end{aligned}$$

Here we have used

$$r^{-1}_{s_{i-1}\cdots s_1}\,\Psi = R_{\epsilon_i-\epsilon_n} \cdots R_{\epsilon_{n-1}-\epsilon_n}\,\Psi\,.$$

Therefore we reach the desired relation (3.2) by using (2.2).

Next we proceed to derive (3.3).

For $1 \le i \le n$, we have

$$r^{-1}_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\langle\,\varphi_{w_i}\,\rangle$$
$$=\int_C x^\lambda \prod_{k=1}^{n} \frac{\left(q^{\frac{1}{2}}\dfrac{ty_k}{x}\right)_\infty}{\left(q^{\frac{1}{2}}\dfrac{y_k}{x}\right)_\infty} \frac{\prod_{k=i+1}^{n}\left(q^{-\frac{1}{2}}\dfrac{ty_k^{-1}}{x}\right)_\infty \prod_{k=1}^{i}\left(q^{\frac{1}{2}}\dfrac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=i}^{n}\left(q^{-\frac{1}{2}}\dfrac{y_k^{-1}}{x}\right)_\infty \prod_{k=1}^{i-1}\left(q^{\frac{1}{2}}\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x} \tag{3.5}$$

and

$$r^{-1}_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\langle\,\varphi_{w_{n+i}}\,\rangle$$
$$=\int_C x^\lambda \frac{\prod_{k=1}^{n-i}\left(q^{\frac{1}{2}}\dfrac{ty_k}{x}\right)_\infty \prod_{k=n-i+1}^{n}\left(q^{\frac{3}{2}}\dfrac{ty_k}{x}\right)_\infty}{\prod_{k=1}^{n-i+1}\left(q^{\frac{1}{2}}\dfrac{y_k}{x}\right)_\infty \prod_{k=n-i+2}^{n}\left(q^{\frac{3}{2}}\dfrac{y_k}{x}\right)_\infty} \prod_{k=1}^{n}\frac{\left(q^{\frac{1}{2}}\dfrac{ty_k^{-1}}{x}\right)_\infty}{\left(q^{\frac{1}{2}}\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x}\,. \tag{3.6}$$

By changing the integration variable such that $x \mapsto q^{-1/2}x$, from (3.5) we have

$$r^{-1}_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\langle\,\varphi_{w_i}\,\rangle$$
$$=q^{-\frac{\lambda}{2}}\int_C x^\lambda \prod_{k=1}^{n} \frac{\left(q\dfrac{ty_k}{x}\right)_\infty}{\left(q\dfrac{y_k}{x}\right)_\infty} \frac{\prod_{k=i+1}^{n}\left(\dfrac{ty_k^{-1}}{x}\right)_\infty \prod_{k=1}^{i}\left(q\dfrac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=i}^{n}\left(\dfrac{y_k^{-1}}{x}\right)_\infty \prod_{k=1}^{i-1}\left(q\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x}$$
$$=q^{-\frac{\lambda}{2}}\langle\, g\varphi_{w_{n+i}}\,\rangle$$

with

$$g = s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_1\cdots s_n) \in W\,.$$

Here we note $g(\epsilon_i) = -\epsilon_{n-i+1}$ for each $1 \le i \le n$.

Similarly, as a result of the change $x \mapsto q^{1/2}x$, from (3.6) we have

$$r^{-1}_{\tau(\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))}\langle\,\varphi_{w_{n+i}}\,\rangle$$
$$=q^{\frac{\lambda}{2}}\int_C x^\lambda \frac{\prod_{k=1}^{n-i}\left(\dfrac{ty_k}{x}\right)_\infty \prod_{k=n-i+1}^{n}\left(q\dfrac{ty_k}{x}\right)_\infty}{\prod_{k=1}^{n-i+1}\left(\dfrac{y_k}{x}\right)_\infty \prod_{k=n-i+2}^{n}\left(q\dfrac{y_k}{x}\right)_\infty} \prod_{k=1}^{n}\frac{\left(\dfrac{ty_k^{-1}}{x}\right)_\infty}{\left(\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x}$$
$$=q^{\frac{\lambda}{2}}\langle\, g\varphi_{w_i}\,\rangle$$

with the same $g \in W$.

As for this $g = s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_1\cdots s_n) \in W$, we have

$$gw_i = w_{n+i}\, s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_2\cdots\ s_{n-1}s_n)\,,$$
$$gw_{n+i} = w_i\, s_n(s_{n-1}s_n)(s_{n-2}s_{n-1}s_n)\cdots(s_2\cdots\ s_{n-1}s_n)$$

for $1 \le i \le n$. These relations lead to

$$g\overline{h}_{w_i} = \overline{h}_{gw_i} = \overline{h}_{w_{n+i}}\,,$$
$$g\overline{h}_{w_{n+i}} = \overline{h}_{gw_{n+i}} = \overline{h}_{w_i}$$

for $1 \le i \le n$.

On the other hand, noting $u = -\lambda\epsilon_1$, we obtain

$$\tau(-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))\overline{h}_{w_i} = q^{\langle\, -\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n),\, -\lambda\epsilon_1\,\rangle}\overline{h}_{w_i} = q^{\frac{\lambda}{2}}\overline{h}_{w_i}\,,$$
$$\tau(-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))\overline{h}_{w_{n+i}} = q^{\langle\, -\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n),\, \lambda\epsilon_{n-i+1}\,\rangle}\overline{h}_{w_{n+i}} = q^{-\frac{\lambda}{2}}\overline{h}_{w_{n+i}}$$

for $1 \le i \le n$.

Combining these relations, we get

$$\begin{aligned}
&\tau(-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))\,\Psi\\
&= \tau(-\frac{1}{2}(\epsilon_1+\cdots+\epsilon_n))\sum_{1\le i\le n}\left\{\langle\,\varphi_{w_i}\,\rangle\,\overline{h}_{w_i} + \langle\,\varphi_{w_{n+i}}\,\rangle\,\overline{h}_{w_{n+i}}\right\}\\
&= \sum_{1\le i\le n}\left\{q^{-\frac{\lambda}{2}}\,\langle\, g\varphi_{w_{n+i}}\,\rangle\, q^{\frac{\lambda}{2}}\overline{h}_{w_i} + q^{\frac{\lambda}{2}}\,\langle\, g\varphi_{w_i}\,\rangle\, q^{-\frac{\lambda}{2}}\overline{h}_{w_{n+i}}\right\}\\
&= \sum_{1\le i\le n}\left\{\,\langle\, g\varphi_{w_{n+i}}\,\rangle\,\overline{h}_{w_i} + \langle\, g\varphi_{w_i}\,\rangle\,\overline{h}_{w_{n+i}}\right\}\\
&= \sum_{1\le i\le n}\left\{\,\langle\, g\varphi_{w_{n+i}}\,\rangle\, g\overline{h}_{w_{n+i}} + \langle\, g\varphi_{w_i}\,\rangle\, g\overline{h}_{w_i}\right\}\\
&= r_g\,\Psi\,.
\end{aligned}$$

At this stage, applying the relation

$$\begin{aligned}
&r_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_n}\,\Psi\\
&= R_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_{n-1}(\alpha_n)}R_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k\cdots s_{n-2}(\alpha_{n-1})}\\
&\times\cdots\times R_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))s_k(\alpha_{k+1})}R_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_n))(\alpha_k)}\\
&\times r_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_{n-1}s_n))}\,\Psi\\
&= R_{2\epsilon_k}R_{\epsilon_k+\epsilon_{k+1}}\cdots R_{\epsilon_k+\epsilon_{n-1}}R_{\epsilon_k+\epsilon_n}\\
&\times r_{(s_n(s_{n-1}s_n)\cdots(s_{k+1}\cdots s_{n-1}s_n))}\,\Psi\,, \qquad (1\le k\le n)
\end{aligned}$$

repeatedly, we finally obtain

$$\begin{aligned}
r_g\,\Psi &= (R_{2\epsilon_1}R_{\epsilon_1+\epsilon_2}\cdots R_{\epsilon_1+\epsilon_n})(R_{2\epsilon_2}R_{\epsilon_2+\epsilon_3}\cdots R_{\epsilon_2+\epsilon_n})\\
&\quad\times\cdots\times(R_{2\epsilon_{n-1}}R_{\epsilon_{n-1}+\epsilon_n})\,R_{2\epsilon_{n-1}}\,\Psi\,.
\end{aligned}$$

Therefore, we reach the desired result (3.3). $\square$

4. Proof of Proposition 3.1

To prove Proposition 3.1, we start by considering the action of $s_i \in W$ on the φ_{w_k}.

Lemma 4.1. (a) *If* $1 \le i \le n-1$, $s_i\,\varphi_{w_k} = \varphi_{w_k}$ *for each* $1 \le k \le 2n$ *such that* $k \ne i,\, i+1,\, 2n-i,\, 2n-i+1$.

(b) $s_n\,\varphi_{w_k} = \varphi_{w_k}$ *for each* $1 \le k \le 2n$ *such that* $k \ne n,\, n+1$.

(c) $s_0\,\varphi_{w_k} = \varphi_{w_k}$ *for each* $1 \le k \le 2n$ *such that* $k \ne 1,\, 2n$.

Proof. These assertions follow from the definition of s_i and φ_{w_k}. □

Moreover we have

Lemma 4.2. (a) *For* $1 \le i \le n-1$;

$$\begin{cases} s_i\varphi_{w_{i+1}} &= a_{\alpha_i}\ \varphi_{w_i} + d_{\alpha_i}\ \varphi_{w_{i+1}}\,, \\ s_i\varphi_{w_i} &= b_{\alpha_i}\ \varphi_{w_i} + c_{\alpha_i}\ \varphi_{w_{i+1}}\,, \end{cases} \tag{4.1}$$

and

$$\begin{cases} s_i\varphi_{w_{2n-i+1}} &= a_{\alpha_i}\ \varphi_{w_{2n-i}} + d_{\alpha_i}\ \varphi_{w_{2n-i+1}}\,, \\ s_i\varphi_{w_{2n-i}} &= b_{\alpha_i}\ \varphi_{w_{2n-i}} + c_{\alpha_i}\ \varphi_{w_{2n-i+1}}\,. \end{cases} \tag{4.2}$$

(b)

$$\begin{cases} s_n\varphi_{w_{n+1}} &= a_{\alpha_n}\ \varphi_{w_n} + d_{\alpha_n}\ \varphi_{w_{n+1}}\,, \\ s_n\varphi_{w_n} &= b_{\alpha_n}\ \varphi_{w_n} + c_{\alpha_n}\ \varphi_{w_{n+1}}\,. \end{cases} \tag{4.3}$$

Proof. By direct calculation or expansion of partial fractions, we find

$$\begin{aligned} &\frac{1-\dfrac{y_{i+1}^{-1}}{x}}{\left(1-t\dfrac{y_i^{-1}}{x}\right)\left(1-t\dfrac{y_{i+1}^{-1}}{x}\right)} \\ &= a_{\alpha_i}\,\frac{1}{1-t\dfrac{y_i^{-1}}{x}} + d_{\alpha_i}\,\frac{1-\dfrac{y_i^{-1}}{x}}{\left(1-t\dfrac{y_{i+1}^{-1}}{x}\right)\left(1-t\dfrac{y_i^{-1}}{x}\right)} \end{aligned} \tag{4.4}$$

and

$$\frac{1}{1-t\dfrac{y_{i+1}^{-1}}{x}} = b_{\alpha_i}\,\frac{1}{1-t\dfrac{y_i^{-1}}{x}} + c_{\alpha_i}\,\frac{1-\dfrac{y_i^{-1}}{x}}{\left(1-t\dfrac{y_{i+1}^{-1}}{x}\right)\left(1-t\dfrac{y_i^{-1}}{x}\right)}\,. \tag{4.5}$$

Multiplying the factor

$$\prod_{j=1}^{i-1}\frac{1-\dfrac{y_j^{-1}}{x}}{1-t\dfrac{y_j^{-1}}{x}}$$

on both sides of each equality (4.4) or (4.5), we get the desired relations (4.1).

While the change of variables $\epsilon_i \mapsto -\epsilon_{i+1}$ and $\epsilon_{i+1} \mapsto -\epsilon_i$ leave α_i unchanged, they produce the following from (4.4) and (4.5):

$$\begin{aligned}&\frac{1-\dfrac{y_i}{x}}{\left(1-t\dfrac{y_i}{x}\right)\left(1-t\dfrac{y_{i+1}}{x}\right)}\\&=a_{\alpha_i}\,\frac{1}{1-t\dfrac{y_{i+1}}{x}}+d_{\alpha_i}\,\frac{1-\dfrac{y_{i+1}}{x}}{\left(1-t\dfrac{y_i}{x}\right)\left(1-t\dfrac{y_{i+1}}{x}\right)}\end{aligned}\tag{4.6}$$

and

$$\frac{1}{1-t\dfrac{y_i}{x}}=b_{\alpha_i}\,\frac{1}{1-t\dfrac{y_{i+1}}{x}}+c_{\alpha_i}\,\frac{1-\dfrac{y_{i+1}}{x}}{\left(1-t\dfrac{y_i}{x}\right)\left(1-t\dfrac{y_{i+1}}{x}\right)}\,.\tag{4.7}$$

Multiplying the factor

$$\prod_{j=i+2}^{n}\frac{1-\dfrac{y_j}{x}}{1-t\dfrac{y_j}{x}}\prod_{j=1}^{n}\frac{1-\dfrac{y_j^{-1}}{x}}{1-t\dfrac{y_j^{-1}}{x}}$$

on both sides of equalities (4.6) and (4.7), we obtain the desired relations (4.2).

Similarly, changing $\epsilon_i \mapsto \epsilon_n$ and $\epsilon_{i+1} \mapsto -\epsilon_n$ induces $\alpha_i \mapsto \alpha_n$ and leads from equalities (4.4) and (4.5) to

$$\begin{aligned}&\frac{1-\dfrac{y_n}{x}}{\left(1-t\dfrac{y_n}{x}\right)\left(1-t\dfrac{y_n^{-1}}{x}\right)}\\&=a_{\alpha_n}\,\frac{1}{1-t\dfrac{y_n^{-1}}{x}}+d_{\alpha_n}\,\frac{1-\dfrac{y_n^{-1}}{x}}{\left(1-t\dfrac{y_n}{x}\right)\left(1-t\dfrac{y_n^{-1}}{x}\right)}\end{aligned}\tag{4.8}$$

and

$$\frac{1}{1-t\dfrac{y_n}{x}}=b_{\alpha_n}\,\frac{1}{1-t\dfrac{y_n^{-1}}{x}}+c_{\alpha_n}\,\frac{1-\dfrac{y_n^{-1}}{x}}{\left(1-t\dfrac{y_n}{x}\right)\left(1-t\dfrac{y_n^{-1}}{x}\right)}\,.\tag{4.9}$$

Multiplying the factor

$$\prod_{j=1}^{n-1}\frac{1-\dfrac{y_j^{-1}}{x}}{1-t\dfrac{y_j^{-1}}{x}}$$

on both sides of equalities (4.8) and (4.9), we get the desired relations (4.3). □

In contrast to the action of s_i for $1 \le i \le n$, the action of s_0 is understood as it acts on the q de Rham cohomology, not on the rational functions.

Lemma 4.3.

$$\begin{cases} q^{\lambda}\langle\, s_0\varphi_{w_1}\,\rangle &= a_{\delta-\theta}\langle\, \varphi_{w_{2n}}\,\rangle + q^{\lambda} d_{\delta-\theta}\langle\, \varphi_{w_1}\,\rangle\,, \\ q^{-\lambda}\langle\, s_0\varphi_{w_{2n}}\,\rangle &= q^{-\lambda} b_{\delta-\theta}\langle\, \varphi_{w_{2n}}\,\rangle + c_{\delta-\theta}\langle\, \varphi_{w_1}\,\rangle\,. \end{cases} \tag{4.10}$$

Proof. Make the change of variables $\epsilon_i \mapsto -\epsilon_1$ and $\epsilon_{i+1} \mapsto \epsilon_1 - \delta$ (i.e. $y_i^{-1} \mapsto y_1\,,\ y_{i+1}^{-1} \mapsto q y_1^{-1}$) in (4.4) and (4.5). Then we have

$$\begin{aligned} &\frac{1 - q\dfrac{y_1^{-1}}{x}}{\left(1 - t\dfrac{y_1}{x}\right)\left(1 - tq\dfrac{y_1^{-1}}{x}\right)} \\ &= a_{\delta-\theta}\,\frac{1}{1 - t\dfrac{y_1}{x}} + d_{\delta-\theta}\,\frac{1 - \dfrac{y_1}{x}}{\left(1 - tq\dfrac{y_1^{-1}}{x}\right)\left(1 - t\dfrac{y_1}{x}\right)} \end{aligned} \tag{4.11}$$

and

$$\frac{1}{1 - tq\dfrac{y_1^{-1}}{x}} = b_{\delta-\theta}\,\frac{1}{1 - t\dfrac{y_1}{x}} + c_{\delta-\theta}\,\frac{1 - \dfrac{y_1}{x}}{\left(1 - tq\dfrac{y_1^{-1}}{x}\right)\left(1 - t\dfrac{y_1}{x}\right)}\,. \tag{4.12}$$

Integration after multiplying the factor

$$\prod_{j=2}^{n}\frac{1 - \dfrac{y_j}{x}}{1 - t\dfrac{y_j}{x}}\prod_{j=1}^{n}\frac{1 - \dfrac{y_j^{-1}}{x}}{1 - t\dfrac{y_j^{-1}}{x}}\,\Phi$$

on both sides of equalities (4.11) and (4.12) gives the following:

$$\begin{aligned} &\int_C x^{\lambda}\frac{\prod_{k=1}^{n}\left(q\dfrac{ty_k}{x}\right)_\infty}{\left(\dfrac{y_1}{x}\right)_\infty\prod_{k=2}^{n}\left(q\dfrac{y_k}{x}\right)_\infty}\frac{\left(q^2\,\dfrac{ty_1^{-1}}{x}\right)_\infty}{\left(q^2\,\dfrac{y_1^{-1}}{x}\right)_\infty}\prod_{k=2}^{n}\frac{\left(q\,\dfrac{ty_k^{-1}}{x}\right)_\infty}{\left(q\,\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x} \\ &= a_{\delta-\theta}\int_C x^{\lambda}\frac{\prod_{k=1}^{n}\left(q\dfrac{ty_k}{x}\right)_\infty}{\left(\dfrac{y_1}{x}\right)_\infty\prod_{k=2}^{n}\left(q\dfrac{y_k}{x}\right)_\infty}\prod_{k=1}^{n}\frac{\left(q\,\dfrac{ty_k^{-1}}{x}\right)_\infty}{\left(q\,\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x} \\ &+ d_{\delta-\theta}\int_C x^{\lambda}\prod_{k=1}^{n}\frac{\left(q\,\dfrac{ty_k}{x}\right)_\infty}{\left(q\,\dfrac{y_k}{x}\right)_\infty}\frac{\left(q^2\,\dfrac{y_1^{-1}}{x}\right)_\infty\prod_{k=2}^{n}\left(q\,\dfrac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=1}^{n}\left(q\,\dfrac{y_k^{-1}}{x}\right)_\infty}\frac{dx}{x} \end{aligned} \tag{4.13}$$

and

$$\int_C x^\lambda \frac{\left(\frac{ty_1}{x}\right)_\infty}{\left(\frac{y_1}{x}\right)_\infty} \prod_{k=2}^n \frac{\left(q\,\frac{ty_k}{x}\right)_\infty}{\left(q\,\frac{y_k}{x}\right)_\infty} \frac{\left(q^2\frac{y_1^{-1}}{x}\right)_\infty \prod_{k=2}^n \left(q\frac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=1}^n \left(q\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x} \tag{4.14}$$

$$= b_{\delta-\theta} \int_C x^\lambda \frac{\prod_{k=1}^n \left(q\frac{ty_k}{x}\right)_\infty}{\left(\frac{y_1}{x}\right)_\infty \prod_{k=2}^n \left(q\frac{y_k}{x}\right)_\infty} \prod_{k=1}^n \frac{\left(q\,\frac{ty_k^{-1}}{x}\right)_\infty}{\left(q\,\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

$$+ c_{\delta-\theta} \int_C x^\lambda \prod_{k=1}^n \frac{\left(q\,\frac{ty_k}{x}\right)_\infty}{\left(q\,\frac{y_k}{x}\right)_\infty} \frac{\left(q^2\frac{y_1^{-1}}{x}\right)_\infty \prod_{k=2}^n \left(q\frac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=1}^n \left(q\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

Here, changing the integration variable such that $x \mapsto qx$, we have

$$\int_C x^\lambda \frac{\prod_{k=1}^n \left(q\frac{ty_k}{x}\right)_\infty}{\left(\frac{y_1}{x}\right)_\infty \prod_{k=2}^n \left(q\frac{y_k}{x}\right)_\infty} \frac{\left(q^2\,\frac{ty_1^{-1}}{x}\right)_\infty}{\left(q^2\,\frac{y_1^{-1}}{x}\right)_\infty} \prod_{k=2}^n \frac{\left(q\,\frac{ty_k^{-1}}{x}\right)_\infty}{\left(q\,\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

$$= q^\lambda \int_C x^\lambda \frac{\prod_{k=1}^n \left(\frac{ty_k}{x}\right)_\infty}{\left(q^{-1}\,\frac{y_1}{x}\right)_\infty \prod_{k=2}^n \left(\frac{y_k}{x}\right)_\infty} \frac{\left(q\,\frac{ty_1^{-1}}{x}\right)_\infty}{\left(q\,\frac{y_1^{-1}}{x}\right)_\infty} \prod_{k=2}^n \frac{\left(\frac{ty_k^{-1}}{x}\right)_\infty}{\left(\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

$$= q^\lambda \langle\, s_0 \varphi_{w_1} \,\rangle$$

and

$$\int_C x^\lambda \prod_{k=1}^n \frac{\left(q\,\frac{ty_k}{x}\right)_\infty}{\left(q\,\frac{y_k}{x}\right)_\infty} \frac{\left(q^2\,\frac{y_1^{-1}}{x}\right)_\infty \prod_{k=2}^n \left(q\,\frac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=1}^n \left(q\,\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

$$= q^\lambda \int_C x^\lambda \prod_{k=1}^n \frac{\left(\frac{ty_k}{x}\right)_\infty}{\left(\frac{y_k}{x}\right)_\infty} \frac{\left(q\frac{y_1^{-1}}{x}\right)_\infty \prod_{k=2}^n \left(\frac{ty_k^{-1}}{x}\right)_\infty}{\prod_{k=1}^n \left(\frac{y_k^{-1}}{x}\right)_\infty} \frac{dx}{x}$$

$$= q^\lambda \langle\, \varphi_{w_1} \,\rangle\,.$$

Therefore, it is seen that (4.13) and (4.14) are equivalent to the desired relations (4.10). $\square$

Next, we consider the asction of W on the $\overline{h}_{w_k}$.

Lemma 4.4. (a) *If* $1 \le i \le n-1$, $\overline{h}_{s_i w_k} = \overline{h}_{w_k}$ *for* $k \ne i, i+1, 2n-i, 2n-i+1$.
(b) $\overline{h}_{s_n w_k} = \overline{h}_{w_k}$ *for* $k \ne n, n+1$.
(c) $\overline{h}_{s_\theta w_k} = \overline{h}_{w_k}$ *for* $k \ne 1, 2n$.

Proof. In the case that $1 \le i \le n-1$, we have $s_i w_k = w_k s_i$ for $1 \le k \le i-1$ or $2n-i+2 \le k \le 2n$, and $s_i w_k = w_k s_{i+1}$ for $i+2 \le k \le 2n-i-1$. These lead to the desired equalities in (a).

In the same way, the relations $s_n w_k = w_k s_n \quad (k \ne n, n+1)$ and $s_\theta w_k = w_k (s_2 \cdots s_{n-1})(s_n \cdots s_2) \quad (k \ne 1, 2n)$ lead to the relations in (b) and (c). □

Next we consider the action of R_{α_i} on the $\overline{h}_{w_k}$:

Lemma 4.5. (a) *If* $1 \le i \le n-1$, $R_{\alpha_i} \overline{h}_{w_k} = \overline{h}_{w_k}$ *for each* $1 \le k \le 2n$ *such that* $k \ne i,\ i+1,\ 2n-i,\ 2n-i+1$.
(b) $R_{\alpha_n} \overline{h}_{w_k} = \overline{h}_{w_k}$ *for each* $1 \le k \le 2n$ *such that* $k \ne n,\ n+1$.
(c) $R_{\delta-\theta} \overline{h}_{w_k} = \overline{h}_{w_k}$ *for each* $2 \le k \le 2n-1$.

Proof. Since $w_k^{-1} \alpha_i = \alpha_i > 0$ for $1 \le k \le i-1$ (then $i \ge 2$) , we have

$$\begin{aligned} R_{\alpha_i} h_{w_k} &= a_{\alpha_i} h_{w_k} + b_{\alpha_i} h_{s_i w_k} , \\ R_{\alpha_i} h_{s_i w_k} &= c_{\alpha_i} h_{s_i w_k} + d_{\alpha_i} h_{w_k} . \end{aligned}$$

These imply

$$R_{\alpha_i} (h_{w_k} + h_{s_i w_k}) = h_{w_k} + h_{s_i w_k} ,$$

following from the relations $a_{\alpha_i} + d_{\alpha_i} = b_{\alpha_i} + c_{\alpha_i} = 1$. Hence, noting $s_i w_k = w_k s_i$, we obtain $R_{\alpha_i} \overline{h}_{w_k} = \overline{h}_{w_k}$. Other cases are similarly derived. □

Lemma 4.6. (a) *For* $1 \le i \le n-1$,

$$\begin{cases} R_{\alpha_i} \overline{h}_{w_i} = a_{\alpha_i} \overline{h}_{w_i} + b_{\alpha_i} \overline{h}_{w_{i+1}} , \\ R_{\alpha_i} \overline{h}_{w_{i+1}} = c_{\alpha_i} \overline{h}_{w_{i+1}} + d_{\alpha_i} \overline{h}_{w_i} , \end{cases} \quad \begin{cases} R_{\alpha_i} \overline{h}_{w_{2n-i}} = a_{\alpha_i} \overline{h}_{w_{2n-i}} + b_{\alpha_i} \overline{h}_{w_{2n-i+1}} , \\ R_{\alpha_i} \overline{h}_{w_{2n-i+1}} = c_{\alpha_i} \overline{h}_{w_{2n-i+1}} + d_{\alpha_i} \overline{h}_{w_{2n-i}} . \end{cases}$$

(b)

$$\begin{cases} R_{\alpha_n} \overline{h}_{w_n} = a_{\alpha_n} \overline{h}_{w_n} + b_{\alpha_n} \overline{h}_{w_{n+1}} , \\ R_{\alpha_n} \overline{h}_{w_{n+1}} = c_{\alpha_n} \overline{h}_{w_{n+1}} + d_{\alpha_n} \overline{h}_{w_n} . \end{cases}$$

(c)

$$\begin{cases} R_{\delta-\theta} \overline{h}_{w_{2n}} = a_{\delta-\theta} \overline{h}_{w_{2n}} + q^{-\lambda} b_{\delta-\theta} \overline{h}_{w_1} , \\ R_{\delta-\theta} \overline{h}_{w_1} = c_{\delta-\theta} \overline{h}_{w_1} + q^{\lambda} d_{\delta-\theta} \overline{h}_{w_{2n}} . \end{cases}$$

Proof. This follows almost immediately from the definitions. □

At this stage, by combination of the above lemmas, we obtain the following:
In case of $1 \le i \le n-1$, we have

$$r_{s_i} \Psi = \sum_{1 \le k \le 2n} \langle\, s_i \varphi_{w_k} \rangle \overline{h}_{s_i w_k} = \{ \sum_{\substack{k \ne i,\, i+1, \\ 2n-i,\, 2n-i+1}} + \sum_{\substack{k = i,\, i+1, \\ 2n-i,\, 2n-i+1}} \} \langle\, s_i \varphi_{w_k} \rangle \overline{h}_{s_i w_k}$$

$$= \sum_{\substack{k \ne i,\, i+1, \\ 2n-i,\, 2n-i+1}} \langle\, \varphi_{w_k} \rangle \overline{h}_{w_k}$$

$$+ \left\{ b_{\alpha_i} \langle\, \varphi_{w_i} \rangle + c_{\alpha_i} \langle\, \varphi_{w_{i+1}} \rangle \right\} \overline{h}_{s_i w_i} + \left\{ a_{\alpha_i} \langle\, \varphi_{w_i} \rangle + d_{\alpha_i} \langle\, \varphi_{w_{i+1}} \rangle \right\} \overline{h}_{s_i w_{i+1}}$$

$$+\left\{b_{\alpha_i}\langle\varphi_{w_{2n-i}}\rangle+c_{\alpha_i}\langle\varphi_{w_{2n-i+1}}\rangle\right\}\overline{h}_{s_iw_{2n-i}}+\left\{a_{\alpha_i}\langle\varphi_{w_{2n-i}}\rangle+d_{\alpha_i}\langle\varphi_{w_{2n-i+1}}\rangle\right\}\overline{h}_{s_iw_{2n-i+1}}$$

$$=\sum_{\substack{k\neq i,\,i+1,\\ 2n-i,\,2n-i+1}}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}$$

$$+\langle\varphi_{w_i}\rangle\left\{b_{\alpha_i}\overline{h}_{w_{i+1}}+a_{\alpha_i}\overline{h}_{w_i}\right\}+\langle\varphi_{w_{i+1}}\rangle\left\{c_{\alpha_i}\overline{h}_{w_{i+1}}+d_{\alpha_i}\overline{h}_{w_i}\right\}$$

$$+\langle\varphi_{w_{2n-i}}\rangle\left\{b_{\alpha_i}\overline{h}_{w_{2n-i+1}}+a_{\alpha_i}\overline{h}_{w_{2n-i}}\right\}+\langle\varphi_{w_{2n-i+1}}\rangle\left\{c_{\alpha_i}\overline{h}_{w_{2n-i+1}}+d_{\alpha_i}\overline{h}_{w_{2n-i+1}}\right\}$$

$$=R_{\alpha_i}\,\Psi.$$

Similarly, in the case $i=n$, we have

$$r_{s_n}\,\Psi=\left\{\sum_{\substack{1\le k\le 2n\\ k\neq n,\,n+1}}+\sum_{k=n,\,n+1}\right\}\langle s_n\varphi_{w_k}\rangle\overline{h}_{s_nw_k}$$

$$=\sum_{k\neq n,n+1}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}$$

$$+\left\{b_{\alpha_n}\langle\varphi_{w_n}\rangle+c_{\alpha_n}\langle\varphi_{w_{n+1}}\rangle\right\}\overline{h}_{s_nw_n}+\left\{a_{\alpha_n}\langle\varphi_{w_n}\rangle+d_{\alpha_n}\langle\varphi_{w_{n+1}}\rangle\right\}\overline{h}_{s_nw_{n+1}}$$

$$=\sum_{k\neq n,n+1}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}$$

$$+\langle\varphi_{w_n}\rangle\left\{b_{\alpha_n}\overline{h}_{w_{n+1}}+a_{\alpha_n}\overline{h}_{w_n}\right\}+\langle\varphi_{w_{n+1}}\rangle\left\{c_{\alpha_n}\overline{h}_{w_{n+1}}+d_{\alpha_n}\overline{h}_{w_n}\right\}$$

$$=R_{\alpha_n}\,\Psi.$$

Finally, if $i=0$, by noting that

$$s_0\overline{h}_{w_1}=q^{\lambda}\overline{h}_{s_\theta w_1}\quad\text{and}\quad s_0\overline{h}_{w_{2n}}=q^{-\lambda}\overline{h}_{s_\theta w_{2n}}\,,$$

we have

$$r_{s_0}\,\Psi=\sum_{\substack{1\le k\le 2n\\ k\neq 1,\,2n}}\langle\varphi_{w_k}\rangle\overline{h}_{s_\theta w_k}+\langle s_0\varphi_{w_1}\rangle q^{\lambda}\overline{h}_{s_\theta w_1}+\langle s_0\varphi_{w_{2n}}\rangle q^{-\lambda}\overline{h}_{s_\theta w_{2n}}$$

$$=\sum_{k\neq 1,\,2n}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}+\langle s_0\varphi_{w_1}\rangle q^{\lambda}\overline{h}_{w_{2n}}+\langle s_0\varphi_{w_{2n}}\rangle q^{-\lambda}\overline{h}_{w_1}$$

$$=\sum_{k\neq 1,\,2n}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}+\left\{c_{\delta-\theta}\langle\varphi_{w_1}\rangle+q^{-\lambda}b_{\delta-\theta}\langle\varphi_{w_{2n}}\rangle\right\}\overline{h}_{w_1}$$

$$+\left\{q^{\lambda}d_{\delta-\theta}\langle\varphi_{w_1}\rangle+a_{\delta-\theta}\langle\varphi_{w_{2n}}\rangle\right\}\overline{h}_{w_{2n}}$$

$$=\sum_{k\neq 1,\,2n}\langle\varphi_{w_k}\rangle\overline{h}_{w_k}+\langle\varphi_{w_1}\rangle\left\{c_{\delta-\theta}\overline{h}_{w_1}+q^{\lambda}d_{\delta-\theta}\overline{h}_{w_{2n}}\right\}$$

$$+\langle\varphi_{w_{2n}}\rangle\left\{a_{\delta-\theta}\overline{h}_{w_{2n}}+q^{-\lambda}b_{\delta-\theta}\overline{h}_{w_1}\right\}$$

$$=R_{\delta-\theta}\,\Psi.$$

This completes the proof of Proposition 3.1. □

5. Macdonald polynomials

Macdonald introduced the q-difference operators [10] to define his orthogonal polynomials associted with root sytems. In the case of a root system of type C_n, the q-difference operator to define such a polynomial is given by

$$E = \sum_{a_1,\dots,a_n=\pm 1} \prod_{1\le i<j\le n} \frac{1-ty_i^{a_i}y_j^{a_j}}{1-y_i^{a_i}y_j^{a_j}} \prod_{1\le i\le n} \frac{1-ty_i^{2a_i}}{1-y_i^{2a_i}} T_{y_i}^{\frac{1}{2}a_i},$$

where

$$(T_{y_i}f)(y_1,\dots,y_n) = f(y_1,\dots,qy_i,\dots,y_n).$$

Its eigenvalue is known to be

$$\begin{aligned} c_\mu &= \sum_{a_1,\dots,a_n=\pm 1} \prod_{j=1}^{n} q^{\frac{1}{2}\lambda_j a_j} t^{\frac{1}{2}(n-j+1)a_j} \\ &= q^{-\frac{1}{2}(\lambda_1+\cdots+\lambda_n)} \prod_{j=1}^{n} (1+t^j q^{\lambda_{n-j+1}}) \end{aligned}$$

with the parameter $\mu = (\lambda_1,\dots,\lambda_n)$ (We consider only the special case corresponding to the condition $t_1 = t_2 = t$).

As for the eigenfunction of the operator E, we easily find the following:

Corollary 5.1. *The sum*

$$\sum_{i=1}^{2n} t^{i-1} \langle\, \varphi_{w_i} \,\rangle \tag{5.1}$$

is a solution of the equation attached to the parameter $(\lambda, 0, \dots, 0)$:

$$E\psi = c_{(\lambda,0,\dots,0)}\psi. \tag{5.2}$$

Proof. This is proven by applying the result of Kato (Theorem 4.6 in [7]) to our Theorem 3.2. □

We next proceed to simplify the sum (5.1).

We note the equality

$$t^{2n} \prod_{j=1}^{n} \frac{\left(1-\dfrac{y_j}{x}\right)\left(1-\dfrac{y_j}{x}\right)}{\left(1-t\dfrac{y_j}{x}\right)\left(1-t\dfrac{y_j^{-1}}{x}\right)} = 1 + (t-1)\left\{ \sum_{j=1}^{2n} t^{i-1}\varphi_{w_i} \right\}, \tag{5.3}$$

which is demonstrated by using the partial fractions.

On the other hand, we have

$$\langle\, \prod_{j=1}^{n} \frac{\left(1-\dfrac{y_j}{x}\right)\left(1-\dfrac{y_j}{x}\right)}{\left(1-t\dfrac{y_j}{x}\right)\left(1-t\dfrac{y_j^{-1}}{x}\right)} \,\rangle = q^\lambda \int_{\mathcal{C}} \Phi = q^\lambda \,\langle\, 1 \,\rangle, \tag{5.4}$$

which is demonstrated by changing the integration variable such that $x \mapsto qx$. Hence, combination of (5.3) and (5.4) gives the relation

$$\sum_{j=1}^{2n} t^{i-1}\langle\, \varphi_{w_i}\,\rangle = \frac{1-q^{\lambda}t^{2n}}{1-t}\int_{\mathcal{C}}\Phi\,.$$

Therefore we reach

Proposition 5.2. *The function $\int_{\mathcal{C}}\Phi$ is a solution to the equation (5.2).*

It should be remarked that this is valid for arbitrary cycle $\mathcal{C}$ and that linearly independent solutions are obtained by choosing several cycles. This situation is similar to that studied in [15].

In case that the parameter μ is from the set of partitions, the eigenfunction of the form

$$P_\mu(y|q,t) = m_\mu + \sum_{\nu<\mu} a_{\mu\;\nu} m_\nu\,,$$

is the Macdonald polynomial for the root system C_n. Here $m_\mu = \sum_{\nu\in W\mu} e^{\nu}$, and $\nu<\mu$ is defined to be $\mu-\nu\in Q^+$ with Q^+ the positive cone of the root lattice.

In our case, to get the Macdonald polynomial, it is enough to consider the case that λ is a positive integer and take the cycle, with the counterclockwise direction, which encircles the sequence of poles such that $y_i, y_iq, y_iq^2, \dots$, for $1\le i\le n$ and $y_i^{-1}, y_i^{-1}q, y_i^{-1}q^2, \dots$, for $1\le i\le n$. This is an integral representaion of the Macdonald polynomial $P_{(\lambda,0,\dots,0)}(y|q,t)$.

Moreover, applying the q-binomial theorem

$$\sum_{m\ge 0}\frac{(a)_m}{(q)_i}z^m = \frac{(az)_\infty}{(z)_\infty} \quad (|z|<1), \qquad (a)_m = \prod_{0\le k\le m-1}(1-aq^k)$$

and the residue calculus to our integral, we obtain an exact expression of the Macdonald polynomial for the root system C_n.

Theorem 5.3.

$$P_{(\lambda,0,\dots,0)}(y|q,t) = \frac{(q)_\lambda}{(t)_\lambda}\sum_{\substack{i_1+\cdots+i_{2n}=\lambda\\ i_1,\dots,i_{2n}\ge 0}}\frac{(t)_{i_1}\cdots(t)_{i_{2n}}}{(q)_{i_1}\cdots(q)_{i_{2n}}}y_1^{i_1-i_{2n}}y_2^{i_2-i_{2n-1}}\cdots y_n^{i_n-i_{n+1}}\,.$$

Remark. We also have a dirct way to obtain the integral representation of the eigenfunction for (5.2). This will appear in a future paper. For the related work, we also refer the reader to [16]

6. Final comment

We finally make a comment on the meaning of our elements φ_{w_i} from the viewpoint of the Hecke algebra.

Set

$$T_i = t + \frac{1-te^{\alpha_i}}{1-e^{\alpha_i}}(s_i-1), \qquad \text{for} \quad 1\le i\le n\,,$$

where α_i is an element of the simple roots and s_i a corresponding generator of the Weyl group W. This is the Lusztig operator associated with the root system C_n (in the special case $t_1 = t_2 = t)$, which satisfies the following:

$$\begin{aligned}
&(T_i - t)(T_i + 1) = 0 \qquad (1 \leq i \leq n)\,,\\
&T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad (1 \leq i \leq n-2)\,,\\
&T_{n-1}T_nT_{n-1}T_n = T_nT_{n-1}T_nT_{n-1}\,,\\
&T_iT_j = T_jT_i \qquad (|i-j| > 2)\,.
\end{aligned}$$

These are the fundamental relations for the Hecke algebra $H(W)$ associated with the root system of type C_n. The action of the Lusztig operator on our φ_{w_i} is given as follows.

Proposition 6.1. *For* $1 \leq k \leq n$;

$$\begin{cases}
T_i\varphi_{w_k} = t\varphi_{w_k}\,, \qquad i \neq k-1,\, k\,,\\
T_{k-1}\varphi_{w_k} = (t-1)\varphi_{w_k} + \varphi_{w_{k-1}}\,,\\
T_k\varphi_{w_k} = t\varphi_{w_{k+1}}\,,
\end{cases}$$

$$\begin{cases}
T_i\varphi_{w_{n+k}} = t\varphi_{w_{n+k}}\,, \qquad i \neq n-k,\, n-k+1\,,\\
T_{n-k+1}\varphi_{w_{n+k}} = (t-1)\varphi_{w_{n+k}} + \varphi_{w_{n+k-1}}\,,\\
T_{n-k}\varphi_{w_{n+k}} = t\varphi_{w_{n+k+1}}\,.
\end{cases}$$

This shows that the vector space $\oplus_{i=1}^{2n}\mathbb{C}\varphi_{w_i}$ gives the representaion of the Hecke algebra $H(W)$ for the C_n type. Moreover, we can also obtain the representation of the affine Hecke algebra in the space of the q de Rham cohomology. See [16] for A_{n-1} case.

In any case, we expect that such a basis attached to the action of the Hecke algebras could be generalized to the case of higher representaions. This is our future problem.

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